A note on Kuratowski's theorem on meagre sets.

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This note was written based on a discussion among Wojciech Bielas, Andrzej Kucharski, Mateusz Kula and Szymon Plewik.

In [1] it was proven that if $\mathcal{F}$ is a partition into meager sets of Čech complete space X of weight $\leqslant 2^\omega$, then there exists a family $\mathcal{A} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{A}$ does not have the Baire property. Following their proof, we provide a slight generalization, see Theorem 1. Instead of Čech completeness we assume that X is Hausdorff and Choquet space; we replace weight by π-weight.

If $\mathcal{B}$ is a family of subsets of a space X, then

$$\mathcal{B}^\omega := \left\{ \bigcap_{n\in\omega} \bigcup \mathcal{S}_n \colon \mathcal{S}_n \subseteq \mathcal{B} \text{ and } |\mathcal{S}_n| < \omega \text{ for all } n \in \omega \right\}.$$

Lemma 1. *Assume that X is a Hausdorff and Choquet space, $\mathcal{B}$ is a π-base with $|\mathcal{B}| \leqslant 2^{\cdot^{\cdot^{\cdot^{2^\omega}}}}$, $\mathcal{F}$ is a partition into meagre sets and $W \subseteq X$ is a G_δ subset that is not nowhere dense. If $\bigcup \mathcal{A}$ has the Baire property for all $\mathcal{A} \subseteq \mathcal{F}$, then there exists $V \subseteq W$, $V \in \mathcal{B}^\omega$ such that $|\{F \in \mathcal{F} \colon F \cap V \neq \emptyset\}| \geqslant 2^\omega$.*

Proof. Take open sets H_n such that $W = \bigcap_{n\in\omega} H_n$. For each finite sequence $s \in 2^{<\omega}$ we will define a set $U_s \in \mathcal{B}$ such that $U_s \subseteq \operatorname{cl} W$ and a sequence $D_s = (D_s^k)_{k\in\omega}$ of nowhere dense sets. We proceed by induction. For $s = \emptyset$, since $\operatorname{int}\operatorname{cl} W \neq \emptyset$, we can pick a set $U_s \in \mathcal{B}$ such that $U_s \subseteq \operatorname{cl} W$ and also put $D_s^k = \emptyset$ for all $k \in \omega$.

Fix $n \in \omega$ and assume that the sets U_s and sequences D_s are defined for all sequences s of length $\leqslant n$. Let s be a sequence of length n. We will define $U_{s^\frown\varepsilon}$ and $D_{s^\frown\varepsilon}$ for $\varepsilon \in \{0, 1\}$. The family

$$\mathcal{I}_s = \{\mathcal{A} \subseteq \mathcal{F} \colon \bigcup \mathcal{A} \cap U_s \text{ is meagre}\}$$

is a sigma-ideal on $\mathcal{F}$ containing all singletons. $\mathcal{I}_s$ is not maximal, because otherwise there would exist a sigma-additive measure on $\mathcal{F}$ and κ-additive measure on some $\kappa \leqslant |\mathcal{F}|$, but

$$\kappa \leqslant |\mathcal{F}| \leqslant |X| \leqslant 2^{2^{|\mathcal{B}|}} \leqslant 2^{\cdot^{\cdot^{2^\omega}}},$$

which contradicts the fact that κ is strongly inaccessible. Hence there exist disjoint families $\mathcal{A}_s^0$ and $\mathcal{A}_s^1$ such that $\mathcal{A}_s^0 \cup \mathcal{A}_s^1 = \mathcal{F}$ and $\mathcal{A}_s^0, \mathcal{A}_s^1 \notin I_s$. Since $\bigcup \mathcal{A}_s^\varepsilon$ has the Baire property, there exist open sets $J_{s^\frown\varepsilon}$ and meagre sets $L_{s^\frown\varepsilon}$ such that

$$\bigcup \mathcal{A}_s^\varepsilon = J_{s^\frown\varepsilon} \Delta L_{s^\frown\varepsilon}$$

Take sequences $D_{s^\frown\varepsilon}$ of nowhere dense sets such that $L_{s^\frown\varepsilon} = \bigcup_{k\in\omega} D^k_{s^\frown\varepsilon}$. Since $(J_{s^\frown\varepsilon} \Delta L_{s^\frown\varepsilon}) \cap U_s = (J_{s^\frown\varepsilon} \cap U_s) \Delta (L_{s^\frown\varepsilon} \cap U_s)$ is not meagre, $J_{s^\frown\varepsilon} \cap U_s$ is a non-empty open set, which is, by inductive hypothesis, contained in $\operatorname{cl} W \subseteq \operatorname{cl} H_n$. Hence also $J_{s^\frown\varepsilon} \cap U_s \cap H_n$ is a non-empty open set. Consequently there exists a non-empty open set $Z_{s^\frown\varepsilon}$ such that

$$Z_{s^\frown\varepsilon} \subseteq J_{s^\frown\varepsilon} \cap U_s \cap H_n \setminus \bigcup_{m<n, k<n} D^k_{s|m}.$$

Choose a non-empty open set $G_{s^\frown\varepsilon}$ according to the winning strategy of the Choquet game for the chain

$$Z_{s|1} \supseteq G_{s|1} \supseteq \cdots \supseteq Z_{s^\frown\varepsilon} \supseteq G_{s^\frown\varepsilon}.$$

Since $\mathcal{B}$ is a π-base, we can find a set $U_{s^\frown\varepsilon} \in \mathcal{B}$ such that

$$U_{s^\frown\varepsilon} \subseteq G_{s^\frown\varepsilon} \subseteq Z_{s^\frown\varepsilon} \subseteq U_s \subseteq \operatorname{cl} W.$$

For any $F \in \mathcal{F}$, if $F \cap U_{s^\frown\varepsilon} \setminus L_{s^\frown\varepsilon} \neq \emptyset$, then $F \in \mathcal{A}_s^\varepsilon$. For a sequence $s \in 2^\omega$ define $K_s = \bigcap_{n\in\omega} U_{s|n}$. Since the sets G_s were chosen according to the Choquet game strategy, K_s is non-empty for each $s \in 2^\omega$. Since for any $s \in 2^\omega$ we have

$$K_s \cap \bigcup_{n\in\omega} L_{s|n} = \emptyset,$$

it follows that for any $F \in \mathcal{F}$, if $F \cap K_s \neq \emptyset \neq F \cap K_{s'}$, then $s = s'$. Consequently, the families $\{F \in \mathcal{F} \colon F \cap K_s \neq \emptyset\}$ are non-empty and disjoint for distinct $s \in 2^\omega$. Therefore

$$|\bigcup_{s\in 2^\omega} \{F \in \mathcal{F} \colon F \cap K_s \neq \emptyset\}| \geqslant 2^\omega.$$

Put

$$V := \bigcap_{n\in\omega} \bigcup_{s\in 2^\omega} U_{s|n}.$$

Then $V \subseteq W$, $V \in \mathcal{B}^\omega$ and $|\{F \in \mathcal{F} \colon F \cap V \neq \emptyset\}| \geqslant 2^\omega$. $\square$

Theorem 1. *Let X be a Hausdorff and Choquet space such that $\pi w(X) \leqslant 2^\omega$. Let $\mathcal{F}$ be a partition of X into meagre sets. Then there exists a family $\mathcal{A} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{A}$ does not have the Baire property.*

Proof. Let $\mathcal{B}$ be a π-base with $|\mathcal{B}| = \pi w(X)$ and let

$$\mathcal{B}^\omega_* = \{V \in \mathcal{B}^\omega \colon |\{F \in \mathcal{F} \colon F \cap V \neq \emptyset\}| \geqslant 2^\omega\}.$$

Then $|\mathcal{B}^\omega_*| \leqslant |\mathcal{B}^\omega| = \kappa \leqslant 2^\omega$. Take an enumeration $\mathcal{B}^\omega_* = \{V_\alpha \colon \alpha < \kappa\}$ and define transfinite sequences $(F^0_\alpha)_{\alpha<\kappa}$ and $(F^1_\alpha)_{\alpha<\kappa}$ such that $F^\varepsilon_\alpha \in \mathcal{F}$, $F^\varepsilon_\alpha \cap V_\alpha \neq \emptyset$, $F^0_\alpha \neq F^1_\alpha$, and $F^\varepsilon_\alpha \notin \bigcup_{\beta<\alpha}\{F^0_\beta, F^1_\beta\}$ for all $\alpha < \kappa$ and $\varepsilon \in \{0,1\}$. Put

$$\mathcal{A}_0 = \bigcup_{\alpha<\kappa}\{F^0_\alpha\} \text{ and } \mathcal{A}_1 = \bigcup_{\alpha<\kappa}\{F^1_\alpha\}.$$

In search for contradiction suppose that $\bigcup \mathcal{A}$ has the Baire property for any $\mathcal{A} \subseteq \mathcal{F}$. Then

$$\bigcup_{\alpha<\kappa} F^0_\alpha = \bigcup \mathcal{A}_0 = G \cup M$$

for some G_δ set G and meagre set M.

Case 1. The set G is meagre. Then $\bigcup \mathcal{A}_0$ is meagre and there exists a dense G_δ subset $W \subseteq X \setminus \bigcup \mathcal{A}_0$. By Lemma 1 there is $\alpha < \kappa$ such that $V_\alpha \subseteq W$. Then

$$\emptyset \neq V_\alpha \cap F^0_\alpha \subseteq V_\alpha \cap \bigcup \mathcal{A}_0 = \emptyset;$$

a contradiction.

Case 2. The set G is non-meagre. By Lemma 1, there is $\alpha < \kappa$ such that $V_\alpha \subseteq G \subseteq \bigcup \mathcal{A}_0$. Then

$$\emptyset \neq V_\alpha \cap F^1_\alpha \subseteq V_\alpha \cap \bigcup \mathcal{A}_1 = \emptyset;$$

a contradiction. □

References

[1] A. Emeryk, R. Frankiewicz, W. Kulpa, *Remarks on Kuratowski's theorem on meager sets*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), no. 6, 493–498.